ISRAEL JOURNAL OF MATHEMATICS 116 (2000), 1–19

FORMS AND BAER ORDERED $*$-FIELDS

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ABSTRACT

It is well known that for a quaternion algebra, the anisotropy of its norm form determines if the quaternion algebra is a division algebra. In case of biquaternion algebra, the anisotropy of the associated Albert form (as defined in [LLT]) determines if the biquaternion algebra is a division ring. In these situations, the norm forms and the Albert forms are quadratic forms over the center of the quaternion algebras; and they are strongly related to the algebraic structure of the algebras. As it turns out, there is a natural way to associate a tensor product of quaternion algebras with a form such that when the involution is orthogonal, the algebra is a Baer ordered $*$-field iff the associated form is anisotropic.

1. Introduction

Let D be a $*$-field, i.e. a division ring with an involution $*$. In D, we denote the set of nonzero symmetric elements by $S(D,*)$. A subset P in $S(D,*)$ is called a Baer ordering if (i) $P+P \subset P$, (ii) $1 \in P$ and for any nonzero $x \in D$, $xPx^* \subset P$, (iii) $P \cup (-P) = S(D,*)$. In the literature, there are other types of orderings defined over $*$-fields; for a reference, see [C_2].

Let F be the center of D and F' be the fixed field of $*$ in F. $(D,*)$ is called trivial if $D = F$ or $(D,*)$ is a standard quaternion algebra. Suppose $(D,*)$ is trivial. If $(D,*)$ admits a Baer ordering P, then $T' = \{\sum x_t x_t^*\colon x_t \in D\}$ is a preordering on F'. Conversely, if T' is a preordering on F', then as pointed out in [L_2, Chapter 14], a T'-normed semiordering (as defined in [L_2, Definition 14.4]) exists. It is clear from the definition of Baer ordering that any normed T'-semiordering on F' is a Baer ordering on $(D,*)$. Let $T = \{\sum x_t x_t^* : x_t \in F\}$. When $D = F$, $T' = T$. Hence $(F,*)$ admits a Baer ordering iff T is a preordering on F'. When $D = \left(\frac{a,b}{F}\right)$ and $*$ is the standard involution, $T' = T + T(-a) +$

Received December 25, 1997 and in revised form November 4, 1998

$T(-b) + T(ab)$. Hence, T' is a preordering iff T is a preordering and the form $\langle 1, -a, -b, ab\rangle$ is T-anisotropic. Thus, when T is a preordering, the anisotropy of the T-form $\langle 1, -a, -b, ab\rangle$ implies the orderability of $(D, *)$. What about the case when $D = \left(\frac{a,b}{F}\right)$ but $*$ is not standard? For $(D, *)$ to admit a Baer ordering, it is still necessary that T is a preordering. In view of the earlier observation, we may ask the following question.

Does there exist a T-form $\phi(D, *)$ over F' such that $(D, *)$ is Baer ordered iff $\phi(D, *)$ is anisotropic over T?

As we will see, the answer is affirmative when D is a quaternion algebra. Moreover, such a result can be extended to the case when D is a tensor product of quaternion algebras with $*$ satisfying certain conditions. Note that up until now, there is no easy way to determine if $(D, *)$ admits a Baer ordering even when $(D, *)$ is a quaternion algebra with an orthogonal involution. In [Le$_2$], it is shown that a $*$-field $(D, *)$ admits a Baer ordering iff $(D, *)$ is Baer formally real. However, it is not easy to determine if $(D, *)$ is Baer formally real in general.

Our investigation on the orderability of quaternion algebras is also motivated by the following longstanding problem raised by Holland [H$_1$]. Does every formally real $*$-field admit a Baer ordering? A $*$-field $(D, *)$ is said to be formally real if $\sum \alpha_i x \alpha_i^* \neq 0$ for any nonzero elements α_i's in D and $x \in S(D, *)$. By using the results mentioned earlier, we see that the answer is affirmative when $(D, *)$ is trivial. Therefore, the next case to be considered is a quaternion algebra with a nonstandard involution. Thus, it is important to find a necessary and sufficient condition for a quaternion algebra to admit a Baer ordering.

2. Notation and preliminary results

From now on, we fix the following notation. $(D, *)$ is a $*$-field with center F and $[D : F]$ is finite. For any subset E in D, we denote $E\backslash\{0\}$ by $\dot{E}$.

To deal with noncommutative $*$-fields, we often make use of $*$-valuations. The notion was first introduced by Holland [H$_2$]. The main purpose then was to lift Baer $*$-orderings from the residue $*$-fields, see [H$_2$, C$_1$]. In the papers [Le$_1$, Le$_2$], $*$-valuations are used to study $*$-fields finite dimensional over their centers as the dimension of the residue $*$-fields over their centers are usually smaller, and that allows us to apply an induction argument.

A valuation v is said to be a $*$-valuation if $v(a) = v(a^*)$ for all $a \in \dot{D}$. As usual, we denote the valuation ring, residue class division ring and value group by $R_v, \bar{D}_v$ and Γ_D respectively. For any element a in R_v, we denote its image in $\bar{D}_v$ by $\bar{a}$. If E is any subset of D, we denote $\{\bar{a}\colon a \in E \cap R_v\}$ by $\bar{E}$ and $v(\dot{E})$

by Γ_E. When v is a $*$-valuation, $*$ induces an obvious involution $\bar{*}$ on $\bar{D}_v$. For any $d \in \dot{D}$, we define the involution $*_d$ such that $*_d(x) = dx^*d^{-1}$ for all $x \in D$. Clearly, v is also a $*$-valuation with respect to $*_d$. Therefore, $*_d$ also induces an involution $\bar{*}_d$ on $\bar{D}_v$ such that for $y = \bar{x}$ in $\bar{D}_v$, $\bar{*}_d(y) = \overline{dx^*d^{-1}}$.

As defined in [Le$_1$], a Baer ordering P is said to be semicompatible with v if $\{a \in D\colon n - aa^* \in P \text{ for some } n \in \mathbb{N}\} \subset R_v$. Moreover, there is a finest nontrivial $*$-valuation v, the order $*$-valuation of P, semicompatible with P. In general, v is not necessarily compatible with P. The key constraint is that for any $x \in P \cap S(D,*)$, $\{\overline{dx^{-1}} : d \in P \text{ and } v(d) = v(x)\}$ need not be a Baer ordering on $(\bar{D}_v, \bar{*}_d)$. In [Le$_2$, Corollary 4.3], a sufficient condition is given to ensure that $\{\overline{dx^{-1}} : d \in P \text{ and } v(d) = v(x)\}$ is a Baer ordering. When that condition fails, v is not compatible with P. Nevertheless, it is still possible to find a Baer ordering compatible with a coarsening of v.

Proposition 2.1: *Let v be a $*$-valuation semicompatible with a Baer ordering P on $(D,*)$. Suppose $[D : F']$ is finite and Γ_D is of finite rank, i.e. Γ_D has only a finite number of convex subgroups. Then there exist a Baer ordering P' and a coarsening v' of v such that P' is compatible with v'. Furthermore, v' can be chosen such that $v'(\dot{D}) \neq v'(\dot{F}')$ if $\Gamma_D \neq \Gamma_{F'}$.*

Proof: Let $S(\Gamma_D) = v(S(D,*))$ and $H = \Gamma_{F'} + 2\Gamma_D$. Clearly, H is a subgroup of Γ_D and $S(\Gamma_D)$ is a union of H-cosets in Γ_D. As $[D : F']$ is finite, $|\Gamma_D/\Gamma_{F'}|$ is also finite. In particular, $|S(\Gamma_D)/\Gamma_{F'}| = k$ is finite. Therefore, there exist $d_1, \dots, d_k \in S(D,*)$ such that $S(\Gamma_D) = \bigcup_{i=1}^k v(d_i) + H$. (Note that we may assume $d_1 = 1$ and $v(d_i) + H \neq v(d_j) + H$ whenever $i \neq j$.) For each coset $\gamma + 2\Gamma_D \in (\Gamma_{F'} + 2\Gamma_D)/2\Gamma_D$, we fix an element $x_\gamma \in F'$ such that $v(x_\gamma) \in \gamma + 2\Gamma_D$. Let $X_v = \{x_\gamma : \gamma + 2\Gamma_D \in (\Gamma_{F'} + 2\Gamma_D)/2\Gamma_D\}$, $Y_v = \{d_1, \dots, d_k\}$ and $A_v = X_v \cdot Y_v$. Clearly, the mapping $\bar{v}$: $A_v \to S(\Gamma_D)/2\Gamma_D$ defined by $\bar{v}(d) = v(d) + 2\Gamma_D$ is bijective.

If $Y_v = \{1\}$, then $A_v \subset F'$. So, for every $d \in A_v$, $\bar{*}_d = \bar{*}$. On the other hand, we conclude from [Le$_1$, Corollary 2.6] that $\bar{P}$ is a Baer ordering on $(\bar{D}_v, \bar{*})$. By [Le$_1$, Proposition 3.3], $\bar{P}$ can be lifted to a Baer ordering compatible with v.

Next, we assume $Y_v \neq \{1\}$. For any $x \in \dot{D}$, we define

$$G(v(x)) = \{\gamma \in \Gamma_D \colon |\gamma| < |v(x) + 2\gamma'| \forall \gamma' \in \Gamma_D\}.$$

Let $d \in Y_v \backslash \{1\}$. As Γ_D is of finite rank, there exists $d' \in d\dot{F}'$ such that $\bigcap_{x \in d\dot{F}'} G(v(x)) = G(v(d'))$. We claim that $v(d') \notin \Gamma_{F'} + G(v(d'))$. Otherwise, there is an element x in $\dot{F}'$ such that $v(xd') \in G(v(d'))$. As $d \in Y_v \backslash \{1\}$, $v(xd') \notin$

$\Gamma_{F'}$. So $v(xd') \neq 0$. But by the definition of $G(v(xd'))$, $G(v(xd')) \subsetneq G(v(d'))$. This contradicts our assumption on d'. We have thus proved our claim.

Note that when we define Y_v, we could replace each d_i by $x_i d_i$ whenever $x_i \in \dot{F}'$. So by the above argument, we may choose $x_i \in \dot{F}'$ such that $v(x_i d_i) \notin G(v(x_i d_i))$ for each i. Then, after each d_i is replaced by $x_i d_i$, we may assume $v(d) \notin \Gamma_{F'} + G(v(d))$ for every $d \in Y_v \backslash \{1\}$.

Let $\Delta = \bigcup_{d \in Y_v \backslash \{1\}} G(v(d))$. As $|Y_v|$ is finite, $\Delta = G(v(d_0))$ for some $d_0 \in Y_v \backslash \{1\}$. As argued above, $v(d_0) \notin \Delta + G(v(d_0))$. Thus, $v(d_0) \notin \Gamma_{F'} + \Delta$.

Define v': $\dot{D} \to \Gamma_D / \Delta$ such that for all $x \in \dot{D}$, $v'(x) = v(x) + \Delta$. As $v(d_0) \notin \Gamma_{F'} + \Delta$, $v'(d_0) \notin v'(\dot{F}')$. By [Le$_2$, Corollary 4.3], $(\bar{D}_{v'}, \bar{*}_d)$ admits a Baer ordering for all $d \in Y_v$. Since for any $x \in A_v$, there exists $d \in Y_v$ such that $\bar{*}_x = \bar{*}_d$, we conclude that $(\bar{D}_{v'}, \bar{*}_d)$ admits a Baer ordering for all $d \in A_v$. For the valuation v', it is clear that the corresponding $A_{v'}$ can be taken as a subset of $Y_{v'}$. Thus by [Le$_1$, Proposition 3.3] again, we get a Baer ordering compatible with v'. ∎

LEMMA 2.2: *Suppose v is compatible with a Baer ordering P on $(D, *)$. Then for any $d \in P$, v is also compatible with the Baer ordering Pd^{-1} on $(D, *_d)$. Here $*_d$ is the involution on D such that $*_d(x) = dx^* d^{-1}$ for all $x \in D$.*

Proof: By [Le$_1$, Lemma 3.1], Pd^{-1} is a Baer ordering on $(D, *_d)$. Clearly, v is also a $*$-valuation on $(D, *_d)$. Lastly, if $x, y \in Pd^{-1}$ and $v(x) > v(y)$, then $v(xd) > v(yd)$ and $xd, yd \in P$. Since v is compatible with P, $(yd - xd) \in P$. Consequently, $y - x \in Pd^{-1}$. ∎

For the rest of this section, we assume $*$ is an involution on F and F' is the fixed field of $*$ in F. If $F \neq F'$, we fix an element $y \in F'$ such that $F = F'[\sqrt{y}]$ and $\sqrt{y}$ is a skew element in $(F, *)$. Furthermore, we assume $T = \{\sum x_t x_t^* : x_t \in F\}$ is a preordering in F'.

LEMMA 2.3: *Let $u_1, u_2, \ldots, u_n$ be elements of F' and $L = F[\sqrt{u_1}, \ldots, \sqrt{u_n}]$. Suppose $\#$ is an F'-automorphism of L that extends $*$ and $\#(\sqrt{u_t}) = (-1)^{\alpha_t}\sqrt{u_t}$ for $t = 1, \ldots, n$. Let L' be the fixed field of $\#$ in L. Then*

$$T(L) := \{\sum x_t x_t^{\#} : x_t \in L\}$$

is a preordering in L' iff $\bigotimes_{t=1}^n \langle 1, (-1)^{\alpha_t} u_t \rangle$ is T-anisotropic. Furthermore, a T-form ϕ over F' is $T(L)$-anisotropic over L' iff $\bigotimes_{t=1}^n \langle 1, (-1)^{\alpha_t} u_t \rangle \otimes \phi$ is T-anisotropic over F'.

Proof: By using an induction argument, it suffices to prove Lemma 2.3 for the case $n = 1$. Note that necessity is obvious in both assertions. Let $\phi = \langle v_1, \ldots, v_s \rangle$ be a T-form over F'. Suppose $\langle 1, (-1)^{\alpha_1} u_1 \rangle \otimes \phi$ is T-anisotropic.

If u_1 is a square in F, then $L = F$, $\# = *$ and $T(L) = T$. As $(-1)^{\alpha_1}u_1 \in T$, our lemma is obvious. Thus, we may assume $\sqrt{u_1} \notin F$. If x_{tl}'s and y_{tl}'s are in F and

$$0 = \sum_{t=1}^{s}\sum_{l}(x_{tl} + y_{tl}\sqrt{u_1})(x_{tl}^* + y_{tl}^*(-1)^{\alpha_1}\sqrt{u_1})v_t,$$

then $0 = \sum_{t=1}^{s}\sum_{l}(x_{tl}x_{tl}^* + y_{tl}y_{tl}^*(-1)^{\alpha_1}u_1)v_t$. Therefore, $\langle 1, (-1)^{\alpha_1}u_1\rangle \otimes \phi$ is T-isotropic. This is a contradiction.

By putting $\phi = \langle 1\rangle$, the calculation above shows that

$$T(L) := \{\sum x_t x_t^{\#} : x_t \in L\}$$

is a preordering in L' whenever $\langle 1, (-1)^{\alpha_1}u_1\rangle$ is T-anisotropic. Moreover, when $T(L)$ is a preordering, the same calculation shows that ϕ is $T(L)$-anisotropic. ∎

LEMMA 2.4: *Let L be as defined in Lemma 2.3. $[L : F] = 2^n$ if $\bigotimes_{t=1}^{n}\langle 1, (-1)^{\beta_t}u_t\rangle$ is T-anisotropic for any $\beta_1, \dots, \beta_n$ in $\{0, 1\}$.*

Proof: Suppose $n = 1$. If $L = F[\sqrt{u_1}] = F$, then $u_1 \in F^2 \cap F'$. So, $u_1 \in (F')^2$ or $u_1 \in y(F')^2$. Note that $u_1 \in y(F')^2$ is possible only when $F \neq F'$. In the former case, $\langle 1, -u_1\rangle$ is T-isotropic. In the latter case, $\langle 1, u_1\rangle$ is T-isotropic. This contradicts our assumption that $\langle 1, \pm u_1\rangle$ are T-anisotropic. Hence, $[L : F] = 2$.

For $n > 1$, we let $K := F[\sqrt{u_1}, \dots, \sqrt{u_{n-1}}]$ and $\#$ be an automorphism of K that extends $*$ and fixes every $\sqrt{u_t}$ for $t = 1, \dots, n-1$. By the induction assumption, $[K : F] = 2^{n-1}$. By Lemma 2.3, $T(K) := \{\sum x_t x_t^{\#} : x_t \in K\}$ is a preordering. As $\bigotimes_{t=1}^{n-1}\langle 1, u_t\rangle \otimes \langle 1, \pm u_n\rangle$ is T-anisotropic, $\langle 1, \pm u_n\rangle$ are $T(K)$-anisotropic by Lemma 2.3. By using the argument for $n = 1$, we see that $[K(\sqrt{u_n}) : K] = 2$. Hence $[L : F] = 2^n$. ∎

3. Tensor product of quaternion algebras

Let D be a tensor product of quaternion algebras and $*$ be an involution on D. Our first goal is to define a form $\phi(D, *)$ such that $(D, *)$ admits a Baer ordering if $\phi(D, *)$ is anisotropic. Unfortunately, such a form does not always exist when $*$ is of the second kind.

Definition 3.1: Let D be a tensor product of quaternion algebras over a field F. We say an involution $*$ is 'nice' if $*$ is an F/F'-involution and there exist $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n \in F'$ such that

$$D = \left(\frac{a_1, b_1}{F}\right) \otimes_F \cdots \otimes_F \left(\frac{a_n, b_n}{F}\right).$$

In that case, we define $\phi(D,*) = \bigotimes_{t=1}^{n}\langle 1, a_t, b_t, -a_tb_t\rangle$.

Note that in Definition 3.1, it is possible that $F = F'$, i.e. $*$ is of the first kind. Clearly, $*$ is 'nice' if $*$ is of the first kind. When D is a quaternion algebra, we conclude from [S, Theorem 11.2] that any involution of the second kind is also 'nice'. However, we do not know if every involution of the second kind is 'nice' when D is not a quaternion algebra. In general, if $*$ is an F/F'-involution and D is a tensor product of quaternion algebras over F, D may not be expressible as a tensor product of $*$-closed quaternion algebras. For an example, see [CW, Section 4]. Nevertheless, our assumption is weaker than the condition that D is expressible as a tensor product of $*$-closed quaternion algebras.

As suggested in the case of a standard involution on a quaternion algebra, we should view $\phi(D,*)$ as a form over a preordering. From now on, we set $T = \{\sum x_ix_i^* : x_i \in F\}$.

Lemma 3.2: *Suppose D is a quaternion algebra and T is a preordering on F'. Then as a T-form over F', $\phi(D,*)$ is uniquely determined if $*$ is of the first kind.*

Proof: Suppose

$$D = \left(\frac{a,b}{F}\right) = \left(\frac{a',b'}{F}\right).$$

It follows from [L_1, Proposition 2.5] that $\langle 1, a, b, -ab\rangle$ and $\langle 1, a', b', -a'b'\rangle$ are isometric as quadratic forms over F. Therefore, they are also T-isometric. ∎

Note that in general, $\phi(D,*)$ depends very much on the choice of a_t's and b_t's when $*$ is of the second kind. So in defining $\phi(D,*)$, we must first fix the choice of a_t's and b_t's.

From now on, we fix the following notation. Let D be a tensor product of quaternion algebras over F and $*$ be a 'nice' involution on D. Therefore, there exist $a_1, \dots, a_n, b_1, \dots, b_n$ in F' such that

$$D = \left(\frac{a_1,b_1}{F}\right) \otimes_F \cdots \otimes_F \left(\frac{a_n,b_n}{F}\right) \quad \text{and} \quad \phi(D,*) = \bigotimes_{t=1}^{n}\langle 1, a_t, b_t, -a_tb_t\rangle.$$

When $*$ is of the second kind, we fix an element $y \in F'$ such that $F = F'[\sqrt{y}]$ and $\sqrt{y}$ is skew. For each $t = 1, \dots, n$, we define $D_t = \left(\frac{a_t,b_t}{F}\right)$. We let i_t, j_t, k_t be elements in D_t such that

$$i_t^2 = a_t, \quad j_t^2 = b_t \quad \text{and} \quad k_t = i_tj_t = -j_ti_t.$$

Clearly, $D_t = F + Fi_t + Fj_t + Fk_t$. As defined earlier, $T = \{\sum x_tx_t^* | x_t \in F\}$.

When we write D as a tensor product of quaternion algebras, it is possible to replace each a_t by ya_t and/or b_t by yb_t. The next result shows that those changes do not concern us in determining the anisotropy of the form $\phi(D,*)$.

LEMMA 3.3: *If a_t is replaced by ya_t and/or b_t is replaced by yb_t, then $\phi(D,*)$ is off by a scalar multiple of an element in F'.*

Proof: For convenience, we will drop the subscripts. It suffices to show that there exist x_1, x_2, x_3 in F' such that

$$x_1 \cdot \langle 1, ay, b, -aby\rangle \cong_T x_2 \cdot \langle 1, a, by, -aby\rangle \cong_T x_3 \cdot \langle 1, ay, by, -ab\rangle \cong_T \langle 1, a, b, -ab\rangle.$$

As $-y \in T$, $\langle 1, ay, b, -aby\rangle \cong_T \langle 1, -a, b, ab\rangle$, $\langle 1, a, by, -aby\rangle \cong_T \langle 1, a, -b, ab\rangle$ and $\langle 1, ay, by, -ab\rangle \cong_T \langle 1, -a, -b, -ab\rangle$. Clearly, we can take $x_1 = b, x_2 = a$ and $x_3 = -ab$. ∎

We are now ready to state our main results.

THEOREM 3.4: *Let D and $*$ be as assumed above. If T is a preordering in F' and $\phi(D,*)$ is T-anisotropic, then $(D,*)$ is a Baer ordered $*$-field.*

THEOREM 3.5: *Suppose the involution $*$ is not symplectic. Then T is a preordering in F and $\phi(D,*)$ is T-anisotropic iff $(D,*)$ is a Baer ordered $*$-field.*

In proving Theorem 3.4 and Theorem 3.5, we often need to deal with valuations on F'. It is desirable that for a valuation $v\colon \dot{F}' \to \Gamma_{F'}$, $v(y)$, $v(a_t)$'s and $v(b_t)$'s satisfy certain conditions.

LEMMA 3.6: *Let v be a valuation compatible with T on F'. Then v extends uniquely to F when $F' \neq F$. Furthermore, if we define $\bar{v}\colon \dot{F}' \to \Gamma_{F'}/2\Gamma_{F'}$ such that for all $x \in \dot{F}'$, $\bar{v}(x) = v(x) + 2\Gamma_{F'}$, then there exist nonnegative integers r, s and elements $a'_1, \ldots, a'_n, b'_1, \ldots, b'_n$ in F' such that*

(i) $D = \left(\frac{a'_1, b'_1}{F}\right) \otimes_F \cdots \otimes_F \left(\frac{a'_n, b'_n}{F}\right)$,

(ii) $\phi(D,*) = x \cdot \bigotimes_{t=1}^{n} \langle 1, a'_t, b'_t, -a'_t b'_t\rangle$ *for some* $x \in F'$,

(iii) $\bar{v}(y) = 0$ *or* $\bar{v}(y) \notin \langle \bar{v}(a'_1), \bar{v}(b'_1), \ldots, \bar{v}(a'_n), \bar{v}(b'_n)\rangle$ *if* $F \neq F'$,

(iv) $v(a'_t) = 0$ *for* $t = 1, \ldots, r+s$ *if* $r + s \geq 1$,

(v) $v(b'_t) = 0$ *for* $t = 1, \ldots, r$ *if* $r \geq 1$,

(vi) $\bar{v}(b'_{r+1}), \ldots, \bar{v}(b'_n)$ *are* $\mathbb{Z}_2$*-independent in* Γ_F/Γ_T *if* $r + 1 \leq n$,

(vii) $\bar{v}(a'_{r+s+1}), \ldots, \bar{v}(a'_n), \bar{v}(b'_{r+1}), \ldots, \bar{v}(b'_n)$ *are* $\mathbb{Z}_2$*-independent in* Γ_F/Γ_T *if* $r + s + 1 \leq n$.

Proof: Suppose $F \neq F'$. Clearly, v extends uniquely to F if $v(y) \notin 2\Gamma_{F'}$. If $v(y) \in 2\Gamma_{F'}$, we may assume $v(y) = 0$. However, $\bar{y}$ is not a square in $\overline{F'}$ as T is compatible with v. Hence, v extends uniquely to F.

Let $H = \langle \bar{v}(a_1), \bar{v}(b_1), \dots, \bar{v}(a_n), \bar{v}(b_n) \rangle$. If $F \neq F'$, $\bar{v}(y) \neq 0$ and $\bar{v}(y) \in H$, then there exists a subgroup H' of index 2 in H such that $H = H' + \langle \bar{v}(y) \rangle$. For each t, we set $u_t = a_t$ if $\bar{v}(a_t) \in H'$ and set $u_t = ya_t$ otherwise. Similarly, we set $w_t = b_t$ if $\bar{v}(b_t) \in H'$ and $w_t = yb_t$ otherwise. Clearly,

$$D = \left(\frac{u_1, w_1}{F}\right) \otimes_F \cdots \otimes_F \left(\frac{u_n, w_n}{F}\right),$$

$\langle \bar{v}(u_1), \bar{v}(w_1), \dots, \bar{v}(u_n), \bar{v}(w_n) \rangle = H'$ and H' does not contain $\bar{v}(y)$. By Definition 3.1 and Lemma 3.3, the new form associated becomes $\bigotimes_{t=1}^n \langle 1, u_t, w_t, -u_t w_t \rangle$ which is T-isometric to $x \cdot \bigotimes_{t=1}^n \langle 1, a_t, b_t, -a_t b_t \rangle$ for a suitable element $x \in F'$. Without loss of generality, we may assume $u_t = a_t$ and $w_t = b_t$ for all t. In particular, we may assume $\bar{v}(y) = 0$ or $\bar{v}(y) \notin H$.

For convenience, we identify i_t with $1 \otimes 1 \cdots \otimes i_t \otimes 1 \cdots \otimes 1$, j_t with $1 \otimes 1 \cdots \otimes j_t \otimes 1 \cdots \otimes 1$ and k_t with $1 \otimes 1 \cdots \otimes k_t \otimes 1 \cdots \otimes 1$. Thus, we may assume i_t, j_t, k_t are elements in D for $t = 1, \dots, n$. Let $B = \{\prod_{t=1}^n \alpha_t : \alpha_t = 1, i_t, j_t \text{ or } k_t\}$. Consider now the F'-algebra

$$D' = \left(\frac{a_1, b_1}{F'}\right) \otimes_{F'} \cdots \otimes_{F'} \left(\frac{a_n, b_n}{F'}\right).$$

We claim that there exist a subset $\{i'_t, j'_t, k'_t : t = 1, \dots, n\}$ of D' and nonnegative integers r, s such that $a'_t = i'^2_t, b'_t = j'^2_t$ are elements in F' for $t = 1, \dots, n$; and the following conditions hold:

(a) $\{\alpha \dot{F}' : \alpha \in B\} = \{(\prod_{t=1}^n \alpha_t)\dot{F}' : \alpha_t = 1, i'_t, j'_t \text{ or } k'_t\}$,
(b) $k'_t = i'_t j'_t = -j'_t i'_t$ for $t = 1, \dots, n$,
(c) $i'_t i'_{t'} = i'_{t'} i'_t$, $i'_t j'_{t'} = j'_{t'} i'_t$, and $j'_t j'_{t'} = j'_{t'} j'_t$ if $1 \leq t' \neq t \leq n$,
(d) $v(a'_t) = 0$ for $t = 1, \dots, r+s$ if $r+s \geq 1$,
(e) $v(b'_t) = 0$ for $t = 1, \dots, r$ if $r \geq 1$,
(f) $\bar{v}(b'_{r+1}), \dots, \bar{v}(b'_n)$ are $\mathbb{Z}_2$-independent in $\Gamma_{F'}/2\Gamma_{F'}$ if $r+1 \leq n$,
(g) $\bar{v}(a'_{r+s+1}), \dots, \bar{v}(a'_n), \bar{v}(b'_{r+1}), \dots, \bar{v}(b'_n)$ are $\mathbb{Z}_2$-independent in $\Gamma_{F'}/2\Gamma_{F'}$ if $r+s+1 \leq n$.

Before we prove our claim, we need to prove some preliminary results. Let $\mathcal{A} = \{x\dot{F}' : x \in B\}$. Clearly, $\mathcal{A}$ can be regarded as a multiplicative elementary 2-abelian group. As in [TW], we define a nondegenerate map $B_{\mathcal{A}}\colon \mathcal{A} \times \mathcal{A} \to \{\pm 1\}$ such that

$$B_{\mathcal{A}}(z_1\dot{F}', z_2\dot{F}')z_1 z_2 \dot{F}' = z_2 z_1 \dot{F}' \quad \text{for every } z_1\dot{F}', z_2\dot{F}' \in \mathcal{A}.$$

For any subgroup $\mathcal{G}$ in $\mathcal{A}$, we define

$$\mathcal{G}^\perp = \{z\dot{F}' \colon B_{\mathcal{A}}(z\dot{F}', z'\dot{F}') = 1 \text{ for all } z'\dot{F}' \in \mathcal{G}\}.$$

Let Δ be the divisible closure of $\Gamma_{F'}$. Note that $\{i_t\dot{F}', j_t\dot{F}' : t = 1, 2, \ldots, n\}$ is a basis for the elementary 2-abelian group $\mathcal{A}$. Therefore, there exists a group homomorphism $w\colon \mathcal{A} \to \Delta/\Gamma_{F'}$ such that for $t = 1, 2, \ldots, n$,

$$w(i_t\dot{F}') = \frac{v(a_t)}{2} + \Gamma_{F'} \quad \text{and} \quad w(j_t\dot{F}') = \frac{v(b_t)}{2} + \Gamma_{F'}.$$

We denote $\ker(w)$ by $\mathcal{K}$. Clearly, $\mathcal{K}^\perp = \{zF' \in \mathcal{A}\colon za\dot{F}' = az\dot{F}' \text{ for all } a\dot{F}' \in \mathcal{K}\}$. There are three possible cases.

Case (i): $\mathcal{K} = \{\dot{F}'\}$. In this case, we set $i'_t = i_t$ and $j'_t = j_t$ for all t. It is straightforward to check that $r = s = 0$ and (a)–(g) hold.

Case (ii): $\mathcal{K} \neq \{\dot{F}'\}$ and $\mathcal{K} = \mathcal{K} \cap \mathcal{K}^\perp$. Let $i'_1\dot{F}' \in \mathcal{K}$. In $\mathcal{K}$, there exists a subgroup $\mathcal{K}'$ of index 2 in $\mathcal{K}$ such that $\{\dot{F}', i'_1\dot{F}'\} \cdot \mathcal{K}' = \mathcal{K}$. As $B_{\mathcal{A}}$ is nondegenerate, $\mathcal{K}'^\perp \supsetneq \mathcal{K}^\perp$. Let $j'_t \in \mathcal{K}'^\perp \backslash \mathcal{K}^\perp$ and $\mathcal{B} = \{\dot{F}', i'_1\dot{F}', j'_1\dot{F}', i'_1 j'_1\dot{F}'\}$. Clearly, $\mathcal{B} \cap \mathcal{B}^\perp = \{\dot{F}'\}$, $\mathcal{A} = \mathcal{B} \cdot \mathcal{B}^\perp$ and $w(\mathcal{B}) + w(\mathcal{B}^\perp) = w(\mathcal{A})$. Moreover, as $\mathcal{K}' \subset \mathcal{B}^\perp$ and $|w(\mathcal{A})| = \frac{|\mathcal{B}|}{2} \cdot \frac{|\mathcal{B}^\perp|}{|\mathcal{K}'|}$, $w(\mathcal{B}) \not\subset w(\mathcal{B}^\perp)$. Hence, $w(\mathcal{B}) \cap w(\mathcal{B}^\perp) = \{\Gamma_{F'}\}$.

Case (iii): $\mathcal{K} \backslash (\mathcal{K} \cap \mathcal{K}^\perp) \neq \emptyset$. Let $i'_1 \in \mathcal{K} \backslash \mathcal{K}^\perp$. As $i'_1\dot{F}' \notin \mathcal{K} \cap \mathcal{K}^\perp$, there exists $j'_1\dot{F}' \in \mathcal{K}$ such that $i'_1 j'_1 = -j'_1 i'_1$. Let

$$\mathcal{B} = \{\dot{F}', i'_1\dot{F}', j'_1\dot{F}', i'_1 j'_1\dot{F}'\}.$$

Clearly, $w(\mathcal{B}) = \{\Gamma_{F'}\}$ and $\mathcal{B} \cap \mathcal{B}^\perp = \{1\}$. As $B_{\mathcal{A}}$ is nondegenerate, $\mathcal{A} = \mathcal{B} \cdot \mathcal{B}^\perp$.

We now prove our claim by induction. Suppose $n = 1$. Clearly, $\{i'_1, j'_1, i'_1 j'_1\}$ defined earlier satisfy (a)–(g). In fact, $r = s = 0$ for Case (i); $r = 0$ and $s = 1$ for Case (ii); and $r = 1, s = 0$ for Case (iii).

Next, we assume $n > 1$. As before, we simply take $r = s = 0$ when $\mathcal{K} = \{\dot{F}'\}$. If we are in Case (ii) or Case (iii), we let $\mathcal{B}$ be as defined in the previous argument. By [TW, Lemma 2.5], we see that

$$D' \cong \left(\frac{a'_1, b'_1}{F'}\right) \otimes_{F'} F'[\mathcal{B}^\perp].$$

We can thus apply induction on $F'[\mathcal{B}^\perp]$ to complete the proof of our claim. Note that $r = 0$ and $s \geq 1$ if Case (ii) happens; and $r \geq 1$ if Case (iii) happens.

Lastly, we show (i)–(vii) are satisfied. By (a), we see that

$$D = \left(\frac{a_1', b_1'}{F}\right) \otimes_F \cdots \otimes_F \left(\frac{a_n', b_n'}{F}\right) \quad \text{and} \quad H = \langle \bar{v}(a_1'), \bar{v}(b_1'), \dots, \bar{v}(a_n'), \bar{v}(b_n') \rangle.$$

Therefore, (i) and (iii) hold. Observe that

$$\bigotimes_{t=1}^{n} \langle 1, a_t, b_t, -a_t b_t \rangle = \bigotimes_{t=1}^{n} \langle 1, i_t^2, j_t^2, k_t^2 \rangle$$

and

$$\bigotimes_{t=1}^{n} \langle 1, {i'}_t^2, {j'}_t^2, {k'}_t^2 \rangle = \bigotimes_{t=1}^{n} \langle 1, a_t', b_t', -a_t' b_t' \rangle.$$

Thus (ii) follows from (a). (iv)–(v) follow easily from (d) and (e). Lastly, observe that $\Gamma_T = 2\Gamma_F = 2\Gamma_{F'} \cup (v(y) + 2\Gamma_{F'})$. Hence, (vi) and (vii) follow from (f), (g) and (iii). ∎

Definition 3.7: Suppose ϕ_t is a T form for $t = 1, 2, \dots, n$. We define

$$\otimes_{t=k}^{k'} \phi_t = \langle 1 \rangle \quad \text{whenever } k > k'.$$

Recall that if v is a valuation compatible with a preordering T on F', $T_v := T \cdot (1 + M_v)$ is a preordering fully compatible with v on F'. For more details, we refer the reader to [L$_2$, Chapter 3].

LEMMA 3.8: *Let v be a valuation compatible with T on F'. Suppose a_t's and b_t's satisfy the conditions (iii)–(vii) in Lemma 3.6.*

(1) $\bigotimes_{t=1}^{n} \langle 1, a_t, b_t, -a_t b_t \rangle$ is T^v-anisotropic iff for any $\alpha_{r+1}, \dots, \alpha_{r+s} \in \{0, 1\}$,

$$\bigotimes_{t=1}^{r} \langle 1, \bar{a}_t, \bar{b}_t, -\bar{a}_t \bar{b}_t \rangle \otimes \bigotimes_{t=r+1}^{r+s} \langle 1, (-1)^{\alpha_t} \bar{a}_t \rangle \text{ is } \bar{T}\text{-anisotropic.}$$

(2) Suppose $\phi(D, *) = \bigotimes_{t=1}^{n} \langle 1, a_t, b_t, -a_t b_t \rangle$ is T-anisotropic. For $t = 1, \dots, n$, D_t is a division ring and v has an extension to D_t. (Recall that $D_t = \left(\frac{a_t, b_t}{F}\right)$.) Furthermore,

$$\bar{D}_t = \begin{cases} \left(\frac{\bar{a}_t, \bar{b}_t}{\bar{F}}\right) & \text{if } 1 \le t \le r, \\ \bar{F}[\bar{i}_t] & \text{if } r+1 \le t \le r+s, \\ \bar{F} & \text{if } r+s+1 \le t \le n. \end{cases}$$

Proof: (1) By condition (iii), we see that

$$\bar{v}(a_{r+s+1}), \dots, \bar{v}(a_n), \bar{v}(b_{r+1}), \dots, \bar{v}(b_n)$$

are $\mathbb{Z}_2$-independent in $\Gamma_{F'}/\Gamma_T$. So (1) follows from [L$_2$, Theorem 4.6].

(2) We first show that D_t is a division ring. Let D'_t be the F'-algebra $\left(\frac{a_t,b_t}{F'}\right)$ in D_t. Since $\langle 1,a_t,b_t,-a_tb_t\rangle$ is T-anisotropic, $\langle a_t,b_t,\ -a_tb_t\rangle$ is anisotropic as a quadratic form over F'. Hence by [L$_1$, Theorem 2.7], D'_t is a division ring. We are done if $F = F'$. Otherwise, we may view D_t as the quotient ring $D'_t[x]/(x^2 - y)D'_t[x]$ where x is an indeterminate that commutes with every element in D'_t. Obviously, $(x^2 - y)D'_t[x]$ is a two-sided ideal. As stated in [Co, p. 532], $D'_t[x]/(x^2 - y)D'_t[x]$ is a division ring if it does not have any zero divisor. By another result stated in [Co, p. 534], $D'_t[x]/(x^2 - y)D'_t[x]$ has no zero divisor if the equation $u^2 = y$ has no root in D'_t. Therefore, D_t is a division ring if the equation $u^2 = y$ has no root in D'_t. Suppose $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in F'$ and $(\alpha_1 + \alpha_2 i_t + \alpha_3 j_t + \alpha_4 k_t)^2 = y$. Then it is easy to see that $\alpha_1^2 + \alpha_2^2 a_t + \alpha_3^2 b_t - \alpha_4^2 a_t b_t = y$. As $-y \in T$, $\langle 1,a_t,b_t,-a_tb_t\rangle$ is then T-isotropic. This is impossible. Therefore, D_t is a division ring.

Suppose $r+s+1 \le t \le n$. Then $v(a_t)+2\Gamma_F$ and $v(b_t)+2\Gamma_F$ are $\mathbb{Z}_2$-independent in $\Gamma_F/2\Gamma_F$. By [TW, Proposition 3.5], v extends to a $*$-valuation totally ramified over F.

If $r+1 \le t \le r+s$, then $v(a_t) = 0$ and $v(b_t) \notin 2\Gamma_F$. By (1), $\langle 1, \pm\bar{a}_t\rangle$ are $\bar{T}$-anisotropic. Since $\bar{T} = \{\sum x_t x_t^* : x_t \in \bar{F}\}$, we can apply Lemma 2.4 to conclude that $\bar{a}_t \notin \bar{F}^2$. By an argument used in the proof of [CW, Theorem 3.9], we see that v extends to a valuation on D_t. Furthermore, the residue $*$-field is $\bar{F}[\bar{i}_t]$.

If $1 \le t \le r$, then $v(a_t) = v(b_t) = 0$. As $\langle 1,a_t,b_t,-a_tb_t\rangle$ is T^v-anisotropic, $\langle 1,\bar{a}_t,\bar{b}_t,-\bar{a}_t\bar{b}_t\rangle$ is $\bar{T}$-anisotropic. By an earlier argument, we see that $\left(\frac{\bar{a}_t,\bar{b}_t}{\bar{F}}\right)$ is a division ring. Apply an argument used in the proof of [CW, Theorem 3.8]; we see that v extends to a valuation totally unramified over F on D_t and $\overline{D_t} = \left(\frac{\bar{a}_t,\bar{b}_t}{\bar{F}}\right)$. ∎

Proof of Theorem 3.4: Suppose T is a preordering and

$$\phi(D,*) = \bigotimes_{t=1}^{n}\langle 1,a_t,b_t,-a_tb_t\rangle$$

is T-anisotropic. Apply [L$_2$, Isotropy Principle 18.2] to the form

$$\bigotimes_{t=1}^{n}\langle 1,a_t,b_t,-a_tb_t\rangle;$$

we get a valuation $v\colon \dot{F}' \to \Gamma_{F'}$ compatible with T such that

(I) either $v(a_t) + \Gamma_T \neq \Gamma_T$ or $v(b_t) + \Gamma_T \neq \Gamma_T$ for some $t = 1,\ldots,n$.

(II) $\bigotimes_{t=1}^{n}\langle 1, a_t, b_t, -a_tb_t\rangle$ is T^v-anisotropic.

By Lemma 3.6, we may assume v is defined on F even when $F \neq F'$. Since v is compatible with T, $\bar{T} = \{\sum x_t x_t^{\bar{*}} : x_t \in \bar{F}\}$ is a preordering on $\overline{F'}$. This fact will be used repeatedly in the subsequent argument. Furthermore, in view of Lemma 3.6, we may assume a_t's and b_t's satisfy the conditions (iii)–(vii) there.

Our strategy is to prove by induction on n that

(i) D is a division ring and v extends to a valuation $\tilde{v}$ on D.

(ii) $(D, *)$ admits a Baer ordering.

By Lemma 3.6 and [CW, Lemma 1.8], the extension $\tilde{v}$ obtained in (i) is in fact a $*$-valuation on D. By [Le$_1$, Proposition 3.3], it suffices to show (i) and that $(\bar{D}_{\tilde{v}}, \bar{*}_d)$ is a Baer ordered $*$-field for every $d \in S(D, *)$.

Suppose $n = 1$. By Lemma 3.8 (2), D is a division ring and v extends to a valuation $\tilde{v}$ on D. Recall that r, s are as defined in Lemma 3.6. By condition (I), we see that $r = 0$. When $s = 0$, the residue $*$-field is $(\bar{F}, \bar{*})$. As $\bar{T}$ is a preordering, $(\bar{F}, \bar{*})$ admits a Baer ordering. For all $d \in S(D, *)$, $\bar{*}_d = \bar{*}$, so $(\bar{F}, \bar{*}_d)$ admits a Baer ordering. In case $s = 1$, the residue $*$-field is $(\bar{F}[\bar{i}_1], \bar{*})$. For any $d \in S(D, *)$, $\bar{*}_d|_{\bar{F}} = \bar{*}|_{\bar{F}}$. As $\bar{T} = \{\sum x_t x_t^{\bar{*}} : x_t \in \bar{F}\}$ and $\langle 1, \pm\bar{a}_1\rangle$ are $\bar{T}$-anisotropic, we conclude from Lemma 2.3 that $(\bar{F}[\bar{i}_1], \bar{*}_d)$ admits a Baer ordering.

Suppose $n > 1$. By Lemma 3.8 (2) and [M, Theorem 1], we see that D is a division ring and v has a unique extension to D if

$$E := \left(\frac{\bar{a}_1, \bar{b}_1}{\bar{F}}\right) \otimes_{\bar{F}} \cdots \otimes_{\bar{F}} \left(\frac{\bar{a}_r, \bar{b}_r}{\bar{F}}\right) \otimes_{\bar{F}} \bar{F}[\bar{i}_{r+1}] \otimes_{\bar{F}} \cdots \otimes_{\bar{F}} \bar{F}[\bar{i}_{r+s}]$$

is a division ring. Here, it is understood that

$$\bar{F}[\bar{i}_{r+1}] \otimes_{\bar{F}} \cdots \otimes_{\bar{F}} \bar{F}[\bar{i}_{r+s}] = \bar{F} \quad \text{if } s = 0$$

and

$$\left(\frac{\bar{a}_1, \bar{b}_1}{\bar{F}}\right) \otimes_{\bar{F}} \cdots \otimes_{\bar{F}} \left(\frac{\bar{a}_r, \bar{b}_r}{\bar{F}}\right) = \bar{F} \quad \text{if } r = 0.$$

By Lemma 3.8 (1) and Lemma 2.3, we see that $\bar{F}[\bar{i}_{r+1}, \ldots, \bar{i}_{r+s}]$ is a field of degree 2^s over $\bar{F}$. In particular, this shows that $\bar{F}[\bar{i}_{r+1}] \otimes_{\bar{F}} \cdots \otimes_{\bar{F}} \bar{F}[\bar{i}_{r+s}]$ can be identified as the field $\bar{F}[\bar{i}_{r+1}, \ldots, \bar{i}_{r+s}]$ which we denote by K. Clearly, we may view

$$E = \left(\frac{\bar{a}_1, \bar{b}_1}{K}\right) \otimes_K \cdots \otimes_K \left(\frac{\bar{a}_r, \bar{b}_r}{K}\right).$$

For any involution $\#$ on K with $\#|_{\bar{F}} = \bar{*}|_{\bar{F}}$, we set $T(\#) = \{\sum x_\alpha x_\alpha^\# : x_\alpha \in K\}$. By Lemma 3.8 (1) and Lemma 2.3, $T(\#)$ is a preordering and

$$\bigotimes_{t=1}^{r} \langle 1, \bar{a}_t, \bar{b}_t, -\bar{a}_t\bar{b}_t\rangle$$

is $T(\#)$-anisotropic.

By (I), we see that $r < n$. So if we set $\# = \bar{*}$, then we can apply an induction assumption to see that $(E, \bar{*})$ is a Baer ordered $*$-field. (Clearly, $\bar{*}$ is still 'nice' on E.) This proves that v extends to a valuation $\tilde{v}$ on D. Furthermore, for any $d \in S(D, *)$, $\bar{*}_d|_{\bar{F}} = \bar{*}|_{\bar{F}}$. So if we set $\# = \bar{*}_d$, we may also apply induction to conclude that $(E, \bar{*}_d)$ is a Baer ordered $*$-field. By our earlier remark, we see that D is a Baer ordered $*$-field. ∎

To prove Theorem 3.5, we assume that P is a Baer ordering on $(D, *)$. First, we claim that we may assume all i_t's and j_t's are symmetric. For $t = 1, \dots, n$, we let $\#_t$ be the F/F'-involution on $\left(\frac{a_t, b_t}{F}\right)$ that fixes i_t and j_t. Note that such involution exists because $a_t, b_t \in F'$ for $t = 1, \dots, n$. As $*$ is not symplectic and $*|_F = (\#_1 \otimes \#_2 \cdots \otimes \#_n)|_F$, $*$ is similar to $\#_1 \otimes \#_2 \cdots \otimes \#_n$. By Lemma 2.2, $(D, *)$ admits a Baer ordering iff $(D, \#)$ admits a Baer ordering. Hence, we may assume $* = \#$. This proves our claim.

As defined in [Le$_1$], there is a finest valuation v_P, the order $*$-valuation of P, that is semicompatible with P. Suppose $v_P(\dot{D}) = v_P(\dot{F'})$. Then we may assume $v_P(i_t) = v_P(j_t) = 0$ for all t. Hence, the residue division ring is a tensor product of quaternion algebras over $\bar{F}$. By [Le$_1$, Corollary 2.11], the residue division ring must then be a quaternion algebra whereas the induced involution must be the standard involution. This is impossible as $\bar{i}_t, \bar{j}_t$ are symmetric elements which are not central in the residue division ring. Therefore, we conclude that $v_P(\dot{D}) \neq v_P(\dot{F'})$.

Unfortunately, v_P may not be compatible with P. By Proposition 2.1, there exists a coarsening $v : \dot{D} \to \Gamma_D$ of v_P such that $\Gamma_D \neq \Gamma_{F'}$ and v is compatible with a Baer ordering P' when F is finitely generated over $\mathbb{Q}$. Thus, we first assume F is finitely generated over $\mathbb{Q}$. As before, we may also assume conditions (iii)–(vii) of Lemma 3.6 hold. Let r, s be as defined in Lemma 3.6. To prove Theorem 3.5, it suffices to show that $\bigotimes_{t=1}^{n} \langle 1, a_t, b_t, -a_t b_t\rangle$ is T^v-anisotropic. In view of Lemma 3.8 (1), we only need to show that for any $\alpha_{r+1}, \dots, \alpha_{r+s}$ in $\{0, 1\}$,

$$\phi' := \bigotimes_{t=1}^{r} \langle 1, \bar{a}_t, \bar{b}_t, -\bar{a}_t\bar{b}_t\rangle \otimes \bigotimes_{t=r+1}^{r+s} \langle 1, (-1)^{\alpha_t}\bar{a}_t\rangle$$

is $\bar{T}$-anisotropic. As before, it is understood that $\bigotimes_{t=1}^{r}\langle 1,\bar{a}_t,\bar{b}_t,-\bar{a}_t\bar{b}_t\rangle = \langle 1\rangle$ when $r=0$ and $\bigotimes_{t=r+1}^{r+s}\langle 1,(-1)^{\alpha_t}\bar{a}_t\rangle = \langle 1\rangle$ when $s=0$.

Recall that in the proof of Theorem 3.4, we have shown that

$$\bar{D}_v = \left(\frac{\bar{a}_1,\bar{b}_1}{K}\right)\otimes_K\cdots\otimes_K\left(\frac{\bar{a}_r,\bar{b}_r}{K}\right)$$

where $K=\bar{F}[\bar{i}_{r+1},\ldots,\bar{i}_{r+s}]$. Again, it is understood that $K=\bar{F}$ if $s=0$ and $\bar{D}_v = K$ if $r=0$.

LEMMA 3.9: *Let r and ϕ' be as defined above.*

(i) *If $r=n$, $*$ is of the second kind, $\bar{F}=\overline{F'}$ and $\bar{*}$ is of the first kind.*

(ii) *If $r\neq n$, then $\bigotimes_{t=r+1}^{r+s}\langle 1,(-1)^{\alpha_t}\bar{a}_t\rangle$ is $\bar{T}$-anisotropic. In particular, ϕ' is $\bar{T}$-anisotropic if $r=0$.*

Proof: Since $[D:F]=[\bar{D}_v:K]$, we conclude that $K=\bar{F}$ and $\Gamma_D=\Gamma_F$. However, as $\Gamma_D\neq\Gamma_{F'}$, $*$ must be of the second kind and $K=\bar{F}=\overline{F'}$. Hence, $\bar{*}$ is of the first kind. This proves (i).

Observe that for $t=r+1,\ldots,r+s$, there exists $d_t\in\{1,j_t\}$ such that $d_t i_t = (-1)^{\alpha_t} i_t d_t$. Set $d=d_{r+1}\cdots d_{r+s}$ if $s\neq 0$. Otherwise, set $d=1$. Clearly, $d\in S(D,*)$. In $\bar{D}_v$, $\bar{*}_d$ fixes $\bar{i}_t$ and $\bar{j}_t$ for $t=1,\ldots,r$. Therefore, $\bar{*}_d$ is either orthogonal or is of the second kind on $\bar{D}_v$. As v is compatible with P', $(\bar{D}_v,\bar{*}_d)$ admits a Baer ordering. In particular, $T(d)=\{\sum x_\alpha x_\alpha^{\bar{*}_d} : x_\alpha\in K\}$ is a preordering. By Lemma 2.3, $\bigotimes_{t=r+1}^{r+s}\langle 1,(-1)^{\alpha_t}\bar{a}_t\rangle$ is $\bar{T}$-anisotropic as $\bar{T}=\{\sum x_t x_t^{\bar{*}} : x_t\in\bar{F}\}$. ∎

Proof of Theorem 3.5: We first assume F is finitely generated over $\mathbb{Q}$. This allows us to use the results we have just proved. We shall prove Theorem 3.5 by induction. Suppose $n=1$. If $r=0$, then by Lemma 3.9 (ii), we see that ϕ' is $\bar{T}$-anisotropic. So $\phi(D,*)$ is T-anisotropic. Note that $r=0$ when $*$ is orthogonal as by assumption of v, $\Gamma_D\neq\Gamma_F$. If $r=1$, we conclude from Lemma 3.9 (i) that $\bar{D}_v = \left(\frac{\bar{a}_1,\bar{b}_1}{\overline{F'}}\right)$ and $\bar{*}$ is orthogonal. Since $\overline{P'}$ is a Baer ordering on $(\bar{D}_v,\bar{*})$ and $\bar{*}$ is orthogonal, our previous argument shows that $\langle 1,\bar{a}_1,\bar{b}_1,-\bar{a}_1\bar{b}_1\rangle$ is T_1-anisotropic where $T_1=\{\sum x_\alpha^2 : x_\alpha\in\overline{F'}\}$. By Lemma 3.6 (iii), we see that $T_1=\bar{T}$. So $\langle 1,\bar{a}_1,\bar{b}_1,-\bar{a}_1\bar{b}_1\rangle$ is $\bar{T}$-anisotropic and hence $\phi(D,*)$ is T-anisotropic. This proves Theorem 3.5 for $n=1$.

Now assume $n>1$. By Lemma 3.9 (ii) again, we may assume $0<r\leq n$. Let d and $T(d)$ be as defined in the proof of Lemma 3.9. Recall that $\bar{*}_d$ is 'nice' and $(\bar{D}_v,\bar{*}_d)$ admits a Baer ordering. If $r<n$, we may apply an induction argument

to conclude that

$$\bigotimes_{t=1}^{r}\langle 1, \bar{a}_t, \bar{b}_t, -\bar{a}_t\bar{b}_t\rangle$$

is $T(d)$-anisotropic. By Lemma 2.3, ϕ' is then $\bar{T}$-anisotropic. So again, $\phi(D,*)$ is T-anisotropic. Since $\Gamma_D \neq \Gamma_F$, $r < n$ when $*$ is orthogonal. Hence, $\phi(D,*)$ is T-anisotropic when $*$ is orthogonal. Finally, we consider the case $r = n$. As before, we see that $\bar{*}$ must be orthogonal. By using a similar argument as in case $n = 1$, we conclude that $\phi' = \phi(\bar{D}_v, \bar{*})$ is $\bar{T}$-anisotropic. So $\phi(D,*)$ is then T-anisotropic.

Lastly, we remove the condition that F is finitely generated over $\mathbb{Q}$. Suppose $\phi(D,*) = \langle z_1, \dots, z_l\rangle$ is T-isotropic. Then

$$\sum(\sum u_{\alpha\beta}u^*_{\alpha\beta})z_\alpha = 0$$

for some $u_{\alpha\beta}$'s in F. Let L be the field obtained by adjoining a_i's, b_i's, $u_{\alpha\beta}$'s and $u^*_{\alpha\beta}$'s to $\mathbb{Q}$. Let

$$E := \left(\frac{a_1, b_1}{L}\right) \otimes_L \cdots \otimes_L \left(\frac{a_n, b_n}{L}\right).$$

Clearly, E is $*$-closed and $P \cap E$ is a Baer ordering on $(E, *|_E)$. As L is finitely generated over $\mathbb{Q}$, our previous argument shows that $\phi(E,*) = \langle z_1, \dots, z_l\rangle$ is T''-anisotropic where $T'' := \{\sum x_\gamma x^*_\gamma : x_\gamma \in L\}$. This contradicts our assumption that $\sum(\sum u_{\alpha\beta}u^*_{\alpha\beta})z_\alpha = 0$. Hence we conclude that $\phi(D,*)$ is T-anisotropic. ∎

Note that in the proof of Theorem 3.5, if $r + s \neq n$, $\phi(D,*)$ is T-anisotropic even when $*$ is symplectic. This follows easily by considering the involution $\#_1 \otimes \#_2 \otimes \cdots \#_{n-1} \otimes *_n$ where $*_n$ is the standard involution. However, the following example shows that $\phi(D,*)$ need not be T-anisotropic in general if $*$ is symplectic.

Example: Let $F = \mathbb{Q}(x,y,z)$ where x,y,z are indeterminates. Obviously, F is formally real and $T := \{\sum t_i^2 : t_i \in F\}$ is a preordering in F. Let v: $\dot{F} \to \mathbb{Z} \times \mathbb{Z}$ be the real valuation such that $v(x) = (0,0), v(y) = (1,0)$ and $v(z) = (0,1)$. Set $a_1 = -(1+x)$, $b_1 = y$, $a_2 = x$ and $b_2 = z$. With respect to v, it is easy to see that $\langle 1, \pm\bar{a}_1\rangle$ and $\langle 1, \pm\bar{a}_2\rangle$ are $\bar{T}$-anisotropic. Therefore,

$$\langle 1, a_1, b_1, -a_1b_1\rangle \quad \text{and} \quad \langle 1, a_2, b_2, -a_2b_2\rangle$$

are T^v-anisotropic. Thus, for $t = 1, 2$, $D_t = \left(\frac{a_t, b_t}{F}\right)$ is a quaternion algebra. By Lemma 3.8, we see that v can be extended to D_1 and D_2. Moreover, the residue fields are respectively $\mathbb{Q}[\sqrt{\bar{a}_1}]$ and $\mathbb{Q}[\sqrt{\bar{a}_2}]$. As $\mathbb{Q}[\sqrt{\bar{a}_1}]\otimes\mathbb{Q}[\sqrt{\bar{a}_2}]$ is a field, we can apply Morandi's result to conclude that $D := D_1 \otimes D_2$ is a division ring and v extends to a valuation on D. For convenience, we also denote the extension by v.

Suppose $*$ is the symplectic involution such that i_1, j_1 are skew and i_2, j_2 are symmetric. As $*$ is of the first kind, v is a $*$-valuation. Let Y_v be as defined in the proof of Proposition 2.1. It is not difficult to see that Y_v can be taken to be $\{1, j_2, i_1k_2, j_1k_2\}$. To see that $(D,*)$ admits a Baer ordering compatible with v, it suffices to show $(\bar{D}_v, \bar{*}_d)$ admits a Baer ordering for every $d \in Y_v$.

By our earlier calculation, $\bar{D}_v = \mathbb{Q}[\bar{i}_1, \bar{i}_2]$. As $\bar{D}_v$ is now a field, $(\bar{D}_v, \bar{*}_d)$ admits a Baer ordering if $\{\sum z_\alpha z_\alpha^{\bar{*}_d} : z_\alpha \in \bar{D}_v\}$ is a preordering. By Lemma 2.3, it suffices to check if the forms

$$\langle 1, (1+\bar{x})\rangle \otimes \langle 1, \pm\bar{x}\rangle \quad \text{and} \quad \langle 1, \pm(1+\bar{x})\rangle \otimes \langle 1, -\bar{x}\rangle$$

are $\bar{T}$-anisotropic. (Note that the first two forms correspond to $d = 1, j_2$ and the second two forms correspond to $d = i_1k_2$ and j_1k_2.) To prove that, we need only to show that there exist three orderings P_1, P_2, P_3 on $\mathbb{Q}(\bar{x})$ such that $\{-\bar{x}, (1+\bar{x})\} \subset P_1$, $\{-\bar{x}, -(1+\bar{x})\} \subset P_2$ and $\{\bar{x}, (1+\bar{x})\} \subset P_3$. The existence of P_3 is obvious. For the construction of P_1 and P_2, we choose real numbers t_1, t_2 which are transcendental over $\mathbb{Q}$ and $t_2 < -1 < t_1 < 0$. Define isomorphism $f_i\colon \mathbb{Q}(\bar{x}) \to \mathbb{R}$ such that $f_i(\bar{x}) = t_i$ for $i = 1, 2$. It is then clear that $P_1 = f_1^{-1}(\{x \in \mathbb{R} | x \geq 0\})$ and $P_2 = f_2^{-1}(\{x \in \mathbb{R} | x \geq 0\})$ are the required orderings.

Note that $\langle 1, a_1, a_2\rangle$ is T-isotropic. Therefore,

$$\langle 1, a_1, b_1, -a_1b_1\rangle \otimes \langle 1, a_2, b_2, -a_2b_2\rangle$$

is also T-isotropic. This shows that Theorem 3.5 is no longer true if $*$ is symplectic. However, it is not difficult to see that in our example,

$$\langle 1, a_1, b_1, -a_1b_1\rangle \otimes \langle 1, -a_2, -b_2, a_2b_2\rangle \quad \text{and} \quad \langle 1, -a_1, -b_1, a_1b_1\rangle \otimes \langle 1, a_2, b_2, -a_2b_2\rangle$$

are T^v-anisotropic. In fact, by modifying the proof of Theorem 3.5, it is possible to prove the following:

THEOREM 3.10: *Let D be as defined before. Suppose $n > 1$ and $*$ is a symplectic involution. Then $(D,*)$ admits a Baer ordering iff T is a preordering in F and there exists a valuation v compatible with T on F' such that for $t = 1, \ldots, n$, $\bigotimes_{l \neq t}\langle 1, a_l, b_l, -a_lb_l\rangle \otimes \langle 1, -a_t, -b_t, a_tb_t\rangle$ is T^v-anisotropic.*

In concluding this section, we state an easy corollary of Theorem 3.5 on SAP fields. For a quadratic form characterization of SAP fields, we refer the reader to [P, Theorem 9.1].

COROLLARY 3.11: *Suppose F is a SAP field and $*$ is an involution of the first kind. Then $(D,*)$ does not admit a Baer ordering unless $n=1$ and $*$ is the standard involution.*

4. A possible counter example

As we have mentioned before, one of our motivations in our study of Baer ordered quaternion algebras is to determine if every formally real $*$-field admits a Baer orderings. Let

$$D=\left(\frac{a,b}{F}\right) \quad \text{and} \quad T=\{\textstyle\sum x_t^2\colon x_t\in F\}.$$

In D, we define $*$ to be the orthogonal involution that fixes i,j. It can be shown that $(D,*)$ is formally real iff

$$ab(x^2a+y^2b)\neq ab+t_1+t_2(x^2a+y^2b) \quad \text{and} \quad 0\neq ab+t_1+t_2(x^2a+y^2b)$$

for any $x,y\in F$ and $t_1,t_2\in T$. Hence, $(D,*)$ is formally real iff

$$ab\neq t_1+(t_2x^2+y^2b^2)a+(t_2y^2+x^2a^2)b$$

and

$$0\neq t_1+(t_2x^2+y^2b^2)a+(t_2y^2+x^2a^2)b.$$

We conjecture that the above condition is not equivalent to the condition that $\langle 1,a,b{-}ab\rangle$ is T-anisotropic. To construct a possible counter example, we propose the following.

A POSSIBLE COUNTER EXAMPLE. Let $x_1,\dots,x_r,y_1,\dots,y_s,z_1,\dots,z_t$, a,b be indeterminates. Let $K=\mathbb{Q}(x_1,\dots,x_r,\ y_1,\dots,y_s,z_1,\dots,z_t,a,b)$ and $F=K[\sqrt{u}]$ where

$$u=ab-(x_1^2+\cdots+x_r^2)-a(y_1^2+\cdots+y_s^2)-b(z_1^2+\cdots+z_t^2).$$

Let $D=\left(\frac{a,b}{F}\right)$ and $*$ be the orthogonal involution that fixes i,j. We first show that D is a division ring.

Let $T(K)=\{\sum\alpha_t^2:\alpha_t\in K\}$. Observe that $\langle 1,a,b,-ab\rangle$ is $T(K)$-anisotropic as there exists a real valuation $v:\dot K\to\mathbb{Z}\times\mathbb{Z}$ such that $v(a)=(1,0)$, $v(b)=(0,1)$ and $v(x_i)=0$ for all i. By Lemma 3.8, we see that $\left(\frac{a,b}{K}\right)$ is a division ring. As in the proof of Lemma 3.8, D is a division ring if we show that $z^2\neq u$ for any $z\in\left(\frac{a,b}{K}\right)$. Suppose $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in K$ and $(\alpha_1+\alpha_2i+\alpha_3j+\alpha_4k)^2=u$. Then

$$\alpha_1^2+\alpha_2^2a+\alpha_3^2b-\alpha_4^2ab=ab-(x_1^2+\cdots+x_r^2)-a(y_1^2+\cdots+y_s^2)-b(z_1^2+\cdots+z_t^2).$$

It follows that $\langle 1, a, b, -ab\rangle$ is then $T(K)$-isotropic. This is impossible. Therefore, D is a division ring.

Note that F is formally real and $T = \{\sum x_t^2 : x_t \in F\}$ is a preordering. Clearly, $\langle 1, a, b, -ab\rangle$ is T-isotropic even though $\langle 1, a, b, -ab\rangle$ is $T(K)$-anisotropic. By Theorem 3.4, $(D, *)$ does not admit a Baer ordering.

Clearly, we may order K such that a, b are positive and

$$x_1, \dots, x_r, y_1, \dots, y_s, z_1, \dots, z_t$$

are infinitesimally small compared with any element in $\mathbb{Q}(a, b)$. In that case u is also positive. Obviously, this ordering can be extended to F. In particular, $\langle 1, a, b\rangle$ is T-anisotropic. Therefore, $(D, *)$ is formally real if

$$ab \neq t_1 + (t_2x^2 + y^2b^2)a + (t_2y^2 + x^2a^2)b$$

for any $x, y \in F$ and $t_1, t_2 \in T$. Apparently, it is different from the condition that $\langle 1, a, b, -ab\rangle$ is T-isotropic. So it is reasonable to believe $(D, *)$ is a possible counter example. Moreover, our example is generic in the sense that if there exists a formally real quaternion algebra which does not admit a Baer ordering, then our constructed example is one of them.

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